MATH 2060 TUT011

§8.4

6. Define the sequence (c_n) and (s_n) inductively by $c_1(x) := 1, s_1(x) := x$, and

$$s_n(x) := \int_0^x c_n(t)dt, \quad c_{n+1}(x) := 1 + \int_0^x s_n(t)dt \quad (*)$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}$. Reason as in the proof of Theorem 8.4.1 to conclude that there exist functions $c : \mathbb{R} \to \mathbb{R}$ and $s : \mathbb{R} \to \mathbb{R}$ such that (j) $c''(x) = c(x)$ and $s''(x) = s(x)$ for all $x \in \mathbb{R}$, and (jj) $c(0) = 1$, $c'(0) = 0$ and $s(0) = 0$, $s'(0) = 1$. Moreover, $c'(x) = s(x)$ and $s'(x) = c(x)$ for all $x \in \mathbb{R}$.

c : hyperbolic cosine , s : hyperbolic sine.

Ans: ① By induction, each c_n, s_n are cts on $\mathbb{R}$, and hence integrable on any bounded intervals.

So they are all well-defined by $(*)$.

By FTC, s_n, c_{n+1} are diff. at every pt. and that

$$s_n'(x) = c_n(x), \quad c_{n+1}'(x) = s_n(x) \quad \forall n \in \mathbb{N}, \forall x \in \mathbb{R}$$

Induction also shows that

$$c_{n+1}(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!}$$

$$s_{n+1}(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}$$

② Let $A > 0$. Take $m > n > 2A$.

If $|x| \le A$, then

$$\begin{aligned}
|c_m(x) - c_n(x)| &= \left| \frac{x^{2n}}{(2n)!} + \frac{x^{2n+2}}{(2n+2)!} \cdots + \frac{x^{2m-2}}{(2m-2)!} \right| \\
&\le \frac{A^{2n}}{(2n)!} + \frac{A^{2n+2}}{(2n+2)!} \cdots + \frac{A^{2m-2}}{(2m-2)!} \\
&= \frac{A^{2n}}{(2n)!} \left[1 + \frac{A^2}{(2n+1)(2n+2)} \cdots + \frac{A^{2m-2n-2}}{(2n+1)\cdots(2m-2)} \right] \\
&\le \frac{A^{2n}}{(2n)!} \left[1 + \left(\frac{A}{2n}\right)^2 + \cdots + \left(\frac{A}{2n}\right)^{2m-2n-2} \right] \\
&\le \frac{A^{2n}}{(2n)!} \cdot \frac{1}{1 - \left(\frac{A}{2n}\right)^2} \\
&\le \frac{A^{2n}}{(2n)!} \cdot \frac{16}{15} \qquad \left(\because \frac{A}{2n} < \frac{1}{4} \right)
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{A^{2n}}{(2n)!} \cdot \frac{16}{15} = 0$, it follows from Cauchy Criterion that (c_n) converges uniformly on $[-A,A]$, where $A>0$ is arbitrary.
In particular, $(c_n(x))$ converges $\forall x \in \mathbb{R}$.
Define $c: \mathbb{R} \to \mathbb{R}$ by $c(x) = \lim_{n\to\infty} c_n(x)$
Since c_n cts and $c_n \rightrightarrows c$ on $[-A,A]$, $\forall A>0$,
so c is cts on $[-A,A]$, $\forall A>0$
$\Rightarrow$ c is cts on $\mathbb{R}$
Also $c_n(0)=1$ $\forall n \in \mathbb{N}$ $\Rightarrow$ $c(0)=1$.

OTOH, if $|x| \le A$ and $m > n \ge 2A$, then

$$|S_n(x) - S_m(x)| = \left|\int_0^x \{c_n(t) - c_m(t)\}\, dt\right|$$
$$\le A \cdot \left(\frac{A^{2n}}{(2n)!} \cdot \frac{16}{15}\right)$$

and so (S_n) converges uniformly on $[-A,A]$.
Define $s: \mathbb{R} \to \mathbb{R}$ by $s(x) = \lim_{n\to\infty} S_n(x)$.
It follows that s is cts on $\mathbb{R}$ and $s(0) = 0$

③ Since $\forall A>0$, 1) $c_n \rightrightarrows c$ on $[-A,A]$
2) $c_n' = S_{n-1} \rightrightarrows s$,
Thm 8.2.3 implies that c is diff. on $[-A,A]$, $\forall A>0$, with

$$c'(x) = s(x) \qquad \forall x \in [-A,A]$$

So, c is diff. on $\mathbb{R}$ with $c'(x) = s(x)$ $\forall x \in \mathbb{R}$.
Similarly, s is diff. on $\mathbb{R}$ with $s'(x) = c(x)$ $\forall x \in \mathbb{R}$

Now, (i) $c''(x) = s'(x) = c(x)$, $s''(x) = c'(x) = s(x)$ $\forall x \in \mathbb{R}$
(ii) $c'(0) = s(0) = 0$, $s'(0) = c(0) = 1$ //

7. Show that the functions c, s in the preceding exercise have derivatives of all orders, and that they satisfy the identity $(c(x))^2 - (s(x))^2 = 1$ for all $x \in \mathbb{R}$. Moreover, they are the unique functions satisfying (j) and (jj). (The functions c, s are called the **hyperbolic cosine** and **hyperbolic sine functions**, respectively.)

Ans: • Since $c' = s$ and $s' = c$,
it follows from induction that c, s have derivatives of all orders.

• Observe that
$$\frac{d}{dx}\left[(c(x))^2 - (s(x))^2\right] = 2c(x)s(x) - 2s(x)c(x) = 0$$
Thus $(c(x))^2 - (s(x))^2$ is a constant fcn whose value
is $(c(0))^2 - (s(0))^2 = 1 - 0 = 1$.

• Let c_1, c_2 be 2 fcns s.t. j) $c_i'' = c_i$
jj) $c_i(0) = 1$, $c_i'(0) = 0$

Let $\varphi = c_1 - c_2$
Then $\varphi''(x) = \varphi(x)$ $\forall x \in \mathbb{R}$, and $\varphi(0) = \varphi^{(k)}(0) = 0$ $\forall k \in \mathbb{N}$.
Fix $x \in \mathbb{R} \setminus \{0\}$. Let $I_x = [0, x]$ if $x > 0$; $[x, 0]$ if $x < 0$.
By Taylor's Thm, $\forall n \in \mathbb{N}$, $\exists\, c_n \in I_x$ s.t.
$$\varphi(x) = \sum_{k=1}^{n-1} \cancel{\frac{\varphi^{(k)}(0)}{k!}}^{\,0} x^k + \frac{\varphi^{(n)}(c_n)}{n!} x^n = \frac{\varphi^{(n)}(c_n)}{n!} x^n$$
Since φ, φ' are cts on I_x, $\exists\, K > 0$ s.t.
$$|\varphi(t)|, |\varphi'(t)| \le K \quad \forall t \in I_x$$
$$\Rightarrow |\varphi^{(n)}(t)| \le K \quad \forall t \in I_x$$
Thus $|\varphi(x)| \le K \frac{|x|^n}{n!}$ $\forall n \in \mathbb{N}$.
Since $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$, we conclude that $\varphi(x) = 0$.
Therefore $\varphi \equiv 0$.
i.e. $c_1(x) = c_2(x)$ $\forall x \in \mathbb{R}$

§9.1

2. Show that if a series is conditionally convergent, then the series obtained from its positive terms is divergent, and the series obtained from its negative terms is divergent.

Ans: Let $\sum a_n$ be a series that is conditionally convergent.
i.e. $\sum a_n$ is convergent, but not absolutely convergent.

Set $p_n = \frac{1}{2}(a_n + |a_n|) = \begin{cases} a_n & \text{if } a_n > 0 \\ 0 & \text{otherwise} \end{cases}$

$q_n = \frac{1}{2}(a_n - |a_n|) = \begin{cases} a_n & \text{if } a_n < 0 \\ 0 & \text{otherwise} \end{cases}$

Then $a_n = p_n + q_n$ and
$|a_n| = p_n - q_n = 2p_n - a_n$

Since $\sum a_n$ is convergent, but $\sum |a_n|$ is divergent,
we must have $\sum p_n$ divergent also
(Otherwise, $\sum |a_n| = 2\sum p_n - \sum a_n$ is convergent)

Note $|a_n| = a_n - 2q_n$.
So $\sum q_n$ is divergent similarly //

6. Find an explicit expression for the nth partial sum of $\sum_{n=2}^{\infty} \ln\left(1 - 1/n^2\right)$ to show that this series converges to $-\ln 2$. Is this convergence absolute?

Ans: The n-th partial sum is

$$S_n := \sum_{k=2}^{n} \ln\left(1 - 1/k^2\right)$$

$$= \sum_{k=2}^{n} \ln\left(\frac{(k-1)(k+1)}{k^2}\right)$$

$$= \sum_{k=2}^{n} \ln(k-1) + \sum_{k=2}^{n} \ln(k+1) - 2\sum_{k=2}^{n} \ln(k)$$

$$= \sum_{k=1}^{n-1} \ln(k) + \sum_{k=3}^{n+1} \ln(k) - 2\sum_{k=2}^{n} \ln(k)$$

$$= \cancel{\ln(1)}^{\,0} - \ln(n-1) + \ln(n+1) - \ln 2$$

$$= \ln\left(\frac{n+1}{n-1}\right) - \ln 2$$

Now $\lim S_n = \lim\left(\ln\left(\frac{n+1}{n-1}\right)\right) - \ln 2$

$$= \ln(1) - \ln 2$$

$$= -\ln 2$$

So the series converges to $-\ln 2$

It is also absolutely convergent since

$$\ln\left(1 - \frac{1}{n^2}\right) < 0 \quad \forall\, n \geq 2$$

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9. If (a_n) is a decreasing sequence of strictly positive numbers and if $\sum a_n$ is convergent, show that $\lim(na_n) = 0$.

10. Give an example of a divergent series $\sum a_n$ with (a_n) decreasing and such that $\lim(na_n) = 0$.

So $\lim(n a_n) = 0$ is only a necessary condition, not a sufficient one

Ans: Let $a_n = \frac{1}{(n+1)\ln(n+1)}$. $n \in \mathbb{N}$.

Then $a_n \geq 0$ and (a_n) is decreasing

with $\lim(n a_n) = 0$.

To see that Σa_n is divergent, we may apply the Integral test.

Let $f(x) := \frac{1}{(x+1)\ln(x+1)}$. for $x \geq 1$.

Then
$$\int_1^n f(x)\,dx = \int_1^n \frac{1}{(x+1)\ln(x+1)}\,dx$$
$$= \int_1^n \frac{d(\ln(x+1))}{\ln(x+1)}$$
$$= \ln(\ln(x+1))\Big|_1^n$$
$$= \ln(\ln(n+1)) - \ln(\ln 2)$$
$$\longrightarrow \infty \text{ as } n \to \infty$$

By Integral test, $\Sigma a_n = \Sigma f(n)$ is divergent //

3. Discuss the convergence or the divergence of the series with nth term (for sufficiently large n) given by

(a) $(\ln n)^{-p}$,
(b) $(\ln n)^{-n}$,
(c) $(\ln n)^{-\ln n}$,
(d) $(\ln n)^{-\ln \ln n}$,
(e) $(n \ln n)^{-1}$,
(f) $(n(\ln n)(\ln \ln n)^2)^{-1}$

Ans: c) nth term test : $a_n \to 0$ (no conclusion)

Root test : $a_n^{\frac{1}{n}} = (\ln n)^{-\frac{\ln n}{n}} \to 1$ (no conclusion)

$\left(\lim_{x \to \infty} \ln (\ln x)^{-\frac{\ln n}{n}} = \lim_{x \to \infty} \frac{(\ln(\ln x)) \ln x}{x} = 0 \right)$

Integral test : $\int_2^\infty (\ln x)^{-\ln x}\, dx = ?$

Comparison test :

$\forall x \geq e^2, \quad x^x \geq (e^2)^x$

$\Rightarrow \forall u \geq e^{e^2}, \quad (\ln u)^{\ln u} \geq u^2$

So $\forall n \geq 2000 > e^{e^2}, \quad a_n = (\ln n)^{-\ln n} \leq \frac{1}{n^2}$

Since $\sum \frac{1}{n^2}$ is convergent $\sum a_n$ is also convergent.

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